1 Countable models

1.1 The omitting types theorem

- **If T is countable and the sets $\Sigma_i(x)$ are not isolated, then T has a model which omits all Σ_i.**
Man bastelt ne Henkin-Theorie mit abzählbar vielen Konstanten so dass jeweils ein $\sigma(x) \in \Sigma(x)$ von c nicht erfüllt wird $\Rightarrow$ aufsteigende Kette, nimm die Vereinigung, tadaa.

1.2 $\aleph_0$-categorical theories

- **Ryll-Nardzewski: A countable complete theory is $\aleph_0$-categorical iff for every n it has only finitely many n-types / formulae up to equivalence modulo T.** Folgt aus:
- **Elementar äquivalente, abzählbare ω-saturierte Strukturen sind isomorph.** (A structure is ω-saturated if it realises all types over finite subsets.)
Man baut ne aufsteigende Kette von endlichen elementaren Abbildungen.
Daraus folgt, dass ω-saturierte Strukturen ω-homogen sind (A structure $\mathfrak{M}$ is *ω-homogenous* if every elementary map $f : A_0 \to M$ with $A_0 \subset M$ finite is (elementarily) extendable to any element in M)
- *Zu Ryll-Nardzewski:* Wenn nur endlich viele Formeln, dann gibt es über endlichen Teilmengen A auch nur endlich viele Formeln in einer Variable, also muss jeder Typ isoliert sein und (weil vollständig) realisiert, also ω-saturiert.
Umgekehrt: Wenn's für irgendein n unendlich viele Formeln mit n Variablen gibt, muss es einen nicht-isolierten Typ geben (weil $S_n(T)$ kompakt ist), der also omitted werden kann.
- *Beispiele für $\aleph_0$-kategorisch:*
  - Infset
  - Die Theorie der unendlichen $\mathbb{F}_q$-Vektorräume (sogar in allen unendlichen Kardinalzahlen, weil eindeutig bestimmt über die Dimension)
  - Random Graph (weil QE und für jedes n nur endlich viele Graphen mit n Knoten)
  - DLO (weil QE und es für jedes n nur endliche viele Möglichkeiten gibt n Elemente anzuordnen)
- A theory is *small* if $S_n(T)$ is at most countable for all $n \in \omega$.
A countable complete theory is small iff it has a countable ω-saturated model.
- **Vaught: A countable complete theory cannot have exactly two countable models.**
(oBdA T small. Es gibt ein ω-saturiertes Modell $\mathfrak{A}$. Ein zweites würde einen nicht-isolierten Typen $p(x)$ omitten, der in $\mathfrak{A}$ von $\bar{a}$ erfüllt wird. Wenn T nicht kathegorisch ist, ist es $\mathsf{Th}(\mathfrak{A}, \bar{a})$ aber auch nicht, also hat es ein drittes abzählbares Modell, das nicht saturiert ist, aber p erfüllt.

1.3 Prime models

T countable+complete

- $\mathfrak{A}$ is a *prime model* of a countable theory T, if it can be elementarily embedded into every model of T.
A structure $\mathfrak{A}$ is *atomic*, if every n-tupel $\bar{a}$ is atomic, i.e. $\mathsf{tp}(\bar{a})$ is isolated in $S_n(\varnothing)$.
A model is prime iff it is countable and atomic.
Prime models are unique up to isomorphism and ω-homogenous.
- **T has a prime model iff the isolated types are dense** (i.e. iff every consistent L-formula belongs to an isolated type.)
Wenn T ein Primmodell hat wird jede konsistente Formel von einem Element darin erfüllt und liegt ergo in dessen Typ. Umgekehrt: $\mathfrak{M}_0$ ist atomar genau dann wenn für alle n die Menge
$$\Sigma_n(x_1, ..., x_n) = \{\neg\varphi(x_1, ..., x_n) \mid \varphi \text{ complete}\}$$
nicht erfüllt wird (wenn ein Tupel das erfüllen würde, wär sein Typ ja nicht isoliert!). Zeige, dass Σ_n nicht isoliert ist, wenn die isolierten n-typen nicht dicht sind.
- T complete.**T small $\Rightarrow$ no binary tree of consistent L-formulae (T countable $\Rightarrow$ converse) $\Rightarrow$ isolated types are dense**
1. Wenn's nen binary tree gäbe würde jeder Pfad (überabzählbar viele) zu irgendnem Typ gehören. (Rückrichtung: Abzählbar viele Formeln, aber überabzählbar viele Typen $\Rightarrow$ Baum basteln, siehe unten)
2. Isolierte Typen nicht dicht $\Rightarrow$ es gibt ne konsistente Formel, die nicht von ner vollständigen Formel impliziert wird. Nachdem die Formel nicht vollständig ist, kann sie in disjunkte Formeln zerlegt werden (es gibt mehrere Typen) die wieder nicht vollständig sein kann $\Rightarrow$ binärer Baum.

2 $\aleph_1$-categorical theories

2.1 Indiscernibles

- **T countable, κ infinite cardinal. T has a model of cardinality κ which realises only countably many types over every countable subset.**
Erweiter zu ner Skolemtheorie, nimm ein Model mit κ vielen Indiscernibles (dank dem Standardlemma). Die von diesen erzeugte Unterstruktur ist elementar (Skolemtheorien haben QE) und von Indiscernibles erzeugte Strukturen erfüllen nur abzählbar viele Typen über abzählbaren Teilmengen (Lemma).

2.2 ω-stable theories

T complete with infinite models

- **T is κ-stable iff T is κ-stable for 1-types**
  Man nehme ein saturiertes Modell und ne Teilmenge $|A| \leq \kappa$ Induktion: Betrachte die Einschränkung $\pi : S_n(A) \to S_1(A)$. Jedes $p \in S_1(A)$ ist der Typ eines Elementes $a \in M$ und $\pi^{-1}(p)$ ist somit bijektiv zu $S_{n-1}(aA)$, also gleiche Kardinalität (nach IV also κ).

- **A countable theory which is categorical in some uncountable ordinal κ is ω-stable**
  Angenommen $S(A)$ überabzählbar für irgendein abzählbares $A \subset \mathfrak{N}$, dann gibt's also überabzählbar viele b_i mit unterschiedlichen Typen über A. Dann gibt's ne $\aleph_1$-Unterstruktur die A und die b_i enthält und davon eine κ-Erweiterung. Die erfüllt dann überabzählbar viele Typen, und es gibt n anderes Modell, das nur abzählbar viele erfüllt.

- **ω-stable theories are totally transcendental; totally transcendental theories are κ-stable for all $\kappa \geq |T|$**
  1. Angenommen es gibt ein Modell mit nem binären Baum in n Variablen, dann hat der Baum nur abzählbar viele Parameter, aber überabzählbar viele Pfade.
  2. Angenommen, es gibt mehr als κ viele n-Typen über irgendnem $|A| = \kappa$. Es gibt höchsten κ viele Formeln, also können höchstens κ viele Typen Formeln enthalten, die nur zu κ vielen Typen gehören.
  $\top$ gehört zu mehr als κ vielen Typen (groß). Jede große Formel gehört also zu irgendwelchen zwei verschiedenen Typen, die nur große Formeln enthalten, also kann ich sie disjunkt aufteilen $\Rightarrow$ binärer Baum.
  $\Rightarrow$ A countable theory is ω-stable iff it is totally transcendental.

- **Elementarily equivalent saturated structures of the same cardinality are isomorphic.**
  Bastel ne aufsteigende Folge von elementaren Abbildungen.

- **A countable theory T is κ-categorical iff all models of cardinality κ are saturated.** Angenommen, T ist κ-kathegorisch (überabzählbar), dann ω-stabil und damit total transzendent, also κ-stabil, dann Lemma.

2.3 Prime extensions

- **If T is totally transcendental, every subset of a model has a constructible prime extension** Wenn toal transzendent, dann kein Baum in jedem Modell, also sind die isolierten Typen über jeder Teilmenge in jedem Modell dicht. Mit Zorn's Lemma kann man ein konstruktibles Modell bauen, das entsprechend prim ist.

2.4 Lachlan's theorem

- *Lachlan's theorem:* Any uncountable model $\mathfrak{M}$ of a totally transcendental theory has arbitrarily large elementary extensions, that omit every countable set of $L(M)$-formulas omitted in $\mathfrak{M}$.
  $\Rightarrow$***Downwards Morley's theorem:*** **Every κ-categorical theory ($\kappa \geq \aleph_1$) is $\aleph_1$-categorical.**
  Angenommen, dass nicht. Dann gibt's ein nicht-saturiertes $\aleph_1$-Modell, also nen typ p über ner abzählbaren Teilmenge, der nicht realisiert wird. T ist total transzendent, also krieg ich mit Lachlan ne κ-Erweiterung, die p omitted, also nicht saturiert, also nicht kathegorisch.

2.5 Vaughtian pairs

- *Vaughts Two-cardinal theorem*: If T has a Vaughtian pair, it has a model $\mathfrak{M}$ of cardinality $\aleph_1$ with a countable $L(M)$-definable subset.
  $\Rightarrow$ **If T is categorical in an uncountable cardinality, it does not have a Vaughtian pair.**
  Wenn überabzählbar kathegorisch, dann $\aleph_1$-kathegorisch. Alle $\aleph_1$-Modelle sind also saturiert, also wird jede Formel entweder von endlich vielen oder überabzählbar vielen erfüllt. Wenn aber Vaughtsches Paar existiert, dann gibt's eben ne Formel mit abzählbar unendlich vielen Erfüllern.

- **A theory without Vaughtian pairs eliminates $\exists^\infty x$**

2.6 Algebraic formulas

- **If $p \in S(A)$ is non-algebraic and $A \subseteq B$, then p has a non-algebraic extension in $S(B)$**
  Wenn p nicht algebraisch ist, ist $p \cup \{\neg\psi(x) \mid \psi(x)$ algebraische $L(B)$-Formel$\}$ endlich erfüllbar. Jeder Typ, der die Menge enthält ist also nicht algebraisch.

- A *pregeometry* (X, cl) is a set X and an operator $\mathsf{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for all $A \subseteq X$ and $a, b \in X$:
  1. $A \subseteq \mathsf{cl}(A)$ (Reflexivity)
  2. $\mathsf{cl}(A) = \bigcup_{A' \subseteq A \text{ finite}} \mathsf{cl}(A')$ (finite character)
  3. $\mathsf{cl}(\mathsf{cl}(A)) = \mathsf{cl}(A)$ (transitivity)
  4. $a \in \mathsf{cl}(Ab) \setminus \mathsf{cl}(A) \Rightarrow b \in \mathsf{cl}(Aa)$ (exchange)

  If X is a universe, acl satisfies 1,2 and 3.

2.7 Strongly minimal sets

T complete with infinite models

- Strong minimality only depends on the type of the parameters, not the actual model.
  Beispiele:
  - Infset
  - Theorie der unendlichen K-Vektorräume
  - ACF_p

  Jeweils, weil mir jede Formel nur endliche Mengen oder deren Komplemente definiert.
- **Every minimal formula is strongly minimal if $\mathfrak{M}$ is ω-saturated or T eliminates $\exists^\infty x$**
  If T is totally transcendental, every infinite definable subset of $\mathfrak{M}^n$ contains a minimal set.
- **If φ is strongly minimal without parameters. then $\mathsf{cl}(A) := \mathsf{acl}^M(A) \cap \varphi(\mathfrak{M})$ is a pregeometry on $\varphi(\mathfrak{M})$**
- A theory is strongly minimal iff over every parameter set there is exactly one non-algebraic type.
- If T is strongly minimal, then:
  - Models are uniquely determined by their dimension.
  - The possible dimensions form an end segment of the cardinals
  - A model is ω-saturated iff its dimension is $\geq \aleph_0$
  - All models are ω-homogenous
- **If T is countable and strongly minimal, it is categorical in all uncountable cardinalities.**
  Zwei Modelle der selben Kardinalität haben die gleiche Dimension. Jede Bijektion zwischen den Basen ist dann elementar und lässt sich zu nem Isomorphismus erweitern.

2.8 The Baldwin-Lachlan-Theorem

- *Baldwin-Lachlan:* **Let κ be uncountable. A countable theory T is κ-categorical iff it is ω-stable and has no Vaughtian pairs. (Proof: Show that there is a strongly minimal formula. Then two models of cardinality κ have the same φ-dimension)**
  Wenn katheogirsch, dann ω-saturiert und kein Vaughtsches Paar.
  Umgekehrt, wenn ω-stabil, dann Total Transzendent, also gibt's ein Primmodell. Es gibt ne minimale $L(M_0)$-Formel φ, und weil kein Vaughtsches Paar existiert wird $\exists^\infty$ eliminiert und die Formel ist strongly minimal. Nimme zwei Modelle M_1, M_2, dann ist M_0 o.B.d.A. ne elementare Unterstruktur und M_1 und M_2 sind minimal Extensions von $M_0 \cup \varphi(M_1)$ (bzw. M_2), weil kein Vuaghtsches Paar. Dann hat $\varphi(M_i)$ Kradinalität κ, also sind die Dimensionen über M_0 gleich, also gibt's ne elementare Abbildung die zu nem Isomorphismus erweiterbar ist.
  $\Rightarrow$ **(Morley:) T is $\aleph_1$-categorical iff it is κ-categorical.**

3 Morley rank

3.1 Saturated structures and the monster

- A structure $\mathfrak{M}$ of cardinality $\kappa \geq \omega$ is *special* if $\mathfrak{M}$ is the union of an elementary chain $\mathfrak{M}_\lambda$ where λ runs over all cardinals less than κ and each $\mathfrak{M}_\lambda$ is λ^+-saturated.
  Two elementarily equivalent special structures of the same cardinality are isomorphic.
- Special structures of cardinality κ are κ^+-universal and strongly $\mathsf{cf}(\kappa)$-homogenous.
- **The monster model is κ-saturated, all models are elementarily embeddable and any elementary bijection between subsets can be extended to an automorphism.**
  Ergibt sich aus Global choice als Vereinigung über ne Klassengroße specialising chain.
  A subclass of the monster is definable (over A) iff it is invariant under all automorphisms which fix A pointwise.

3.2 Morley rank

T complete

- Morley rank:
  - $\mathsf{MR}\varphi \geq 0$ if φ is consistent.
  - $\mathsf{MR}\varphi \geq \beta+1$ if there is an infinite family of pairwise inconsistent formulae with morley rank $\geq \beta$ implying φ
  - $\mathsf{MR}\varphi \geq \lambda$ (for λ limit ordinal) if $\mathsf{MR}\varphi \geq \beta$ for all $\beta < \lambda$

$$\mathsf{MR}\varphi = \begin{cases} -\infty & \text{if } \varphi \text{ is inconsistent} \\ \infty & \text{if } \mathsf{MR}\varphi \geq \alpha \text{ for all } \alpha \in \mathbf{On} \\ \max\{\alpha \mid \mathsf{MR}\varphi \geq \alpha\} & \text{otherwise} \end{cases}$$

$\Rightarrow \mathsf{MR}\varphi = 0$ gdw φ is consistent and algebraic.
$\mathsf{MR}(T) = \mathsf{MR}(x \doteq x)$
$\mathsf{MR}(\varphi \vee \psi) = \max\{\mathsf{MR}\varphi, \mathsf{MR}\psi\}$
The morley rank of a type is the minimal morley rank of its formulae.

- *Beispiel:* in Infset hat $\varphi(x) = x \doteq a$ Rang 0 und $\varphi(x, y) = x \doteq a$ Rang 1.
- $\varphi \sim_\alpha \psi$ (α-equivalence) if $\varphi \Delta \psi$ has rank less than α.
  A formula is α-*strongly minimal* if it has rank α and for any formula ψ implying φ, either ψ or $\varphi \wedge \neg\psi$ has rank less than α.
- **Each formula of rank α is equivalent to a disjunction of finitely many pairwise disjoint α-strongly minimal formulae (the *components* of φ).** The components are unique up to α-equivalence.
  Wenn φ nicht schon α-strongly minimal ist gibt's also eine Formel ψ von Rang α, so dass ψ und $(\neg\psi \wedge \varphi)$ beide φ implizieren und Rang α haben. Die lassen sich dann wieder zerlegen ... dann hätte φ aber Rang $> \alpha$. Für eindeutigkeit: Wenn ψ α-strongly minimal ist und φ impliziert, kann ψ in die Formeln $\psi \wedge \varphi_i$ zerlegt werden, eine davon muss α-äquivalent zu ψ sein.
- The *Morley degree* is the number of components. The morley degree of a consistent algebraic formula is the number of realisations.
  φ is strongly minimal iff $\mathsf{MR}\varphi = \mathsf{MD}\varphi = 1$
  If a type has Morley rank α, its Morley degree is the minimal degree of its formulae with rank α.
- **T is totally transcendental iff each formula has Morley rank**
  Wenn ne Formel keinen Morleyrang hat, kann ich sie in zwei Formeln zerlegen, die auch keinen Morleyrang hat, also binärer Baum.
  Umgekehrt, wenn ich nen binären Baum hab, hat keine Formel Morleyrang: Falls doch, hat irgendein φ_s minimalen Rang und Grad. Dann müssten φ_{s0} und φ_{s1} aber den selben Rang, aber niedrigeren Grad haben.
- Der Morley-Grad eines Typs in $S(A), A \subseteq B$ ist die Summe der Morley-Grade aller Erweiterungen von p in $S(B)$ mit gleichem Morleyrang.
  $\Rightarrow$ Jeder Typ hat auf jeder Obermenge mindestens eine und höchstens Morley-Grad viele Erweiterungen von gleichem Morleyrang.

3.3 Countable models of $\aleph_1$-categorical theories

T countable $\aleph_1$-categorical theory

- The φ-dimension of N over M does not depend on φ; it is the maximal length of an elementary chain

$$M = M_0 \underset{\neq}{\prec} M_1 \underset{\neq}{\prec} \ldots \underset{\neq}{\prec} M_n = N$$

- *Baldwin-Lachlan:* Let M_0 be the prime model of T uncountably categorical. For any cardinal $m \geq \dim_\varphi(M_0)$ there is a unique model M with $\dim_\varphi(M) = m$. These models are pairwise non-isomorphic.

3.4 Computation of Morley rank

T countable, complete

- φ strongly minimal, defined over B and $a_1, ..., a_n$ realisations. Then $\mathsf{MR}(a_1, ..., a_n/B) = \dim_\varphi(a_1, ..., a_n/B)$
  $\Rightarrow \varphi(x)$ strongly minimal and $\psi(x_1, ..., x_n)$ defined over B and $\psi \to \varphi(x_i)$ for all i. Then $\mathsf{MR}(\psi) = \max\{\dim_\varphi(\overline{a}/B) \mid \models \psi(\overline{a})\}$.
  $\Rightarrow$ The class $\left\{\overline{b} \mid \psi(x_1, ..., x_n, \overline{b}) = k\right\}$ is definable.
- Sei K ein Körper. Der Morleyrang eines Tupels $\overline{a}$ aus dem Abschluss von K entspricht dem Transzendenzgrad von $\overline{a}$ über K.
- If φ is strongly minimal, defined over B and a, b algebraic over $\varphi(\mathfrak{C}) \cup B$, then $\mathsf{MR}(ab/B) = \mathsf{MR}(a/B) + \mathsf{MR}(b/aB)$.

4 Simple theories

4.1 Dividing and forking

- $\varphi(x, b)$ *divides* over A (with respect to k) if there is a countable sequence b_i of realisations ot $\mathtt{tp}(b/A)$ such that the sequence $(\varphi(x, b_i))$ is k-inconsistent.
  A set of formulae divides, if it implies a dividing formula.
  A set of formulae divides iff a conjunction of some of its formulae divides.

- *Example:* DLO, the formula $b_1 < x < b_2$ divides over the empty set.
  The type $\{x > a \mid a \in \mathbb{Q}\}$ does not divide over the empty set.
  Zyklische Ordnung: $\mathsf{cyc}(a, x, b)$ dividet über der leeren Menge, (für $a_i = i, b_i = i + 1$ paarweise inkonsistent).
  Der einzige Typ (QE!) über der leeren Menge forkt, weil er z.B. $\mathsf{cyc}(-1, x, 2) \vee \mathsf{cyc}(1, x, 0)$ impliziert.

- If $a \notin \mathsf{acl}(A)$ then $\mathsf{tp}(a/Aa)$ divides over A
  If a set $\pi(x)$ is consistent and defined over $\mathsf{acl}(A)$, then π does not divide over A
  The set $\pi(x, b)$ divides over A iff there is an infinite sequence b_i of indiscernibles over A with $\mathsf{tp}(b_0/A) = \mathsf{tp}(b/A)$ and $\bigcup \pi(x, b_i)$ is inconsistent

- If $\mathsf{tp}(a/B)$ does not divide over $A \subseteq B$ and $\mathsf{tp}(c/Ba)$ does not divide over Aa, then $\mathsf{tp}(ac/B)$ does not divide over A.

- A set of formulae π *forks* over A if it implies a disjunction of formulae, each dividing over A.
  Dividing implies forking.
  If $p \in S(B)$ forks over A then 1. **(Non-forking is closed)** there is some $\varphi \in p$ such that any type in $S(B)$ containing φ forks over A and 2. **(Finite character)** there is a finite subset B_0 such that $p \restriction AB_0$ forks over A.
  If π is finitely satisfiable in A then it does not fork over A.
  If π is defined over B and does not fork over $A \subseteq B$ it can be extended to some $p \in S(B)$ which does not fork over A.

4.2 Simplicity

T countable and complete

- A formula $\varphi(x, y)$ has the *tree property* (w.r.t. k) if there is a tree of parameters $(a_s)_{s \in {}^{<\omega}\omega}$, such that every branch is consistent and for every $s \in {}^{<\omega}\omega$ the set $\{\varphi(x, a_{si}) \mid i \in \omega\}$ is k-inconsistent
  A theory is *simple* if no formula has the tree property. Totally transcendental theories are simple.

- For a complete theory, the following are equivalent:
  - T is simple
  - **For all $p \in S_n(B)$ there is some $A \subseteq B$ with $|A| \leq |T|$ such that p does not divide over A (*Local character*)**
  - There is some κ such that for all models M and $p \in S_n(M)$ there is some $A \subseteq M$ with $|A| \leq \kappa$ such that p does not divide over A

  ⇒ **Let T simple and $p \in S(A)$, then p does not fork over A.**
  ⇒ **If T is simple, every type over A has a non-forking extension to any B containing A (*existence*).**

- $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$ if for every finite tuple $\overline{a}$ from A the type $\mathsf{tp}(\overline{a}/BC)$ does not fork over C.
  A sequence $(a_i)_{i \in I}$ is called
  - *independent over A* if $a_i \mathop{\smile\!\!\!\!\!\!|}_{A} \{a_j \mid j < i\}$ for all i
  - a *Morley sequence over A* if it is independent and indiscernible over A
  - a *Morley sequence in $p(x)$ over A* if it is a Morley sequence over A consisting of realisations of p.

  Let q be a global type invariant over A. Any sequence $(b_i)_{i \in I}$ where each b_i realises $q \restriction A \cup \{b_j \mid j < i\}$ is a Morley sequence over A.

- If $p \in S(B)$ does not fork over A, there is an infinite Morley sequence in p over A which is indiscernible over B. If T is simple, for every $p \in S(A)$ there is an infinite Morley sequence in p over A.
  Let T simple and $\pi(x, y)$ a partial type over A, $(b_i)_{i \in \omega}$ an infinite Morley sequence over A and $\bigcup \pi(x, b_i)$ consistent. Then $\pi(x, b_0)$ does not divide over A.

- Let T simple, then $\pi(x, b)$ divides over A iff it forks over A.
  Let T simple, then independence is symmetric, monotone and transitive, i.e.:

  $$A \mathop{\smile\!\!\!\!\!\!|}_{B} C \Leftrightarrow C \mathop{\smile\!\!\!\!\!\!|}_{B} A \qquad B \subseteq C \subseteq D \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_{B} D \leftrightarrow A \mathop{\smile\!\!\!\!\!\!|}_{B} C \wedge A \mathop{\smile\!\!\!\!\!\!|}_{C} D$$

  ⇒ The independence of a sequence does not depend on the ordering.
  Let T be simple and $\mathcal{I}$ an infinite Morley sequence over A. If $\mathcal{I}$ is indiscernible over Ac, then $c \mathop{\smile\!\!\!\!\!\!|}_{A} \mathcal{I}$

4.3 The independence theorem

T simple

- For a set A we write $\mathsf{nc}_A(a, b)$ if a and b start an infinite sequence of indiscernibles over A
  A formula $\theta(x, y)$ is called *thick* if there are no infinite antichains, i.e. sequences $(c_i)_{i \in \omega}$ where $\neg\theta(c_i, c_j)$ for all $i < j < \omega$.
  (T arbitrary) $\mathsf{nc}_A(a, b)$ iff $\models \theta(a, b)$ for all thick θ defined over A
  ⇒ If a, b have the same type over a model M, there is some c such that $\mathsf{nc}_M(a, c)$ and $\mathsf{nc}_M(c, b)$.

- *Independence theorem*: Let b, c have the same type over some model M and $B \underset{M}{\perp} C, b \underset{M}{\perp} B, c \underset{M}{\perp} C$. Then there is some d with $\mathsf{tp}(d/B) = \mathsf{tp}(b/B)$ and $\mathsf{tp}(d/C) = \mathsf{tp}(c/C)$ such that $d \underset{M}{\perp} BC$.

  ⇒*Kim-Pillay*: Let T be complete and $a \overset{0}{\underset{A}{\perp}} B$ a relation between finite tuples a and sets A, B invariant under automorphisms with the following properties:

  1. *Monotonicity and transitivity:* $a \overset{0}{\underset{A}{\perp}} BC \Leftrightarrow a \overset{0}{\underset{A}{\perp}} B \wedge a \overset{0}{\underset{AB}{\perp}} C$
  2. *Symmetry*: $a \overset{0}{\underset{A}{\perp}} b \Leftrightarrow b \overset{0}{\underset{A}{\perp}} a$
  3. *Finite character:* $a \overset{0}{\underset{A}{\perp}} B$ if $a \overset{0}{\underset{A}{\perp}} b$ for all finite tuples $b \in B$
  4. *Local character:* There is a cardinal κ such that for all a and B there exists $B_0 \subseteq B$ of of cardinality less than κ such that $a \overset{0}{\underset{B_0}{\perp}} B$
  5. *Existence*: For all a, A, C there is a' such that $\mathsf{tp}(a'/A) = \mathsf{tp}(a/A)$ and $a' \overset{0}{\underset{A}{\perp}} C$
  6. *Independence over models*: For every model M and a, b, a', b' with $\mathsf{tp}(a'/M) = \mathsf{tp}(b'/M)$ and $a' \overset{0}{\underset{M}{\perp}} a, b' \overset{0}{\underset{M}{\perp}} b, a \overset{0}{\underset{M}{\perp}} b$ there is some $c$such that $\mathsf{tp}(c/Ma) = \mathsf{tp}(a'/Ma), \mathsf{tp}(c/Mb) = \mathsf{tp}(b'/Mb)$ and $c \overset{0}{\underset{M}{\perp}} ab$

  then T is simple and $\overset{0}{\perp} = \perp$

4.4 Lascar strong types

T complete

- Let A be any set of parameters. The group $\mathsf{Aut}_f(\mathfrak{C}/A)$ of *Lascar strong automorphisms* of $\mathfrak{C}$ over A is the group generated by all $\mathsf{Aut}(\mathfrak{C}/M)$ where the M are models containing A. Two tuples a and b have the same Lascar strong type over A if $\alpha(a) = b$ for some $\alpha \in \mathsf{Aut}_f(\mathfrak{C}/A)$. We denote this by $\mathsf{Lstp}(a/A) = \mathsf{Lstp}(b/A)$.
  Tuples a and b have the same Lascar strong type over A iff there is a sequence $a = b_0, b_1, ..., b_n = b$ such that for all for all $i < n, b_i$ and b_{i+1} have the same type over some model containing A.

- *Independence theorem*: Let T simple and $\mathsf{Lstp}(b/A) = \mathsf{Lstp}(c/A), B \underset{A}{\perp} C, b \underset{A}{\perp} B, c \underset{A}{\perp} C$. Then there is some d such that $d \underset{A}{\perp} BC, \mathsf{Lstp}(d/B) = \mathsf{Lstp}(b/B)$ and $\mathsf{Lstp}(d/C) = \mathsf{Lstp}(c/C)$

5 Stable theories

A theory is κ-stable if there are only κ-many types over any parameter set of size κ. A theory is stable if it is κ-stable for some κ.

5.1 Heirs and coheirs

T complete

- Let p be a type over a model M of T and $q \in S(B)$ an extension of p to $B \supset M$. Then q is an *heir* of p if for every $L(M)$-formula $\varphi(x, y)$ such that $\varphi(x, b) \in q$ for some $b \in B$ there is some $m \in M$ with $\varphi(x, m) \in p$.
  q is a coheir of p if q is finitely satisfiable in M.
  $\mathsf{tp}(a/Mb)$ is an heir of $\mathsf{tp}(a/M)$ iff $\mathsf{tp}(b/Ma)$ is a coheir of $\mathsf{tp}(b/M)$.
  Definitionen anwenden und Variablen in $\varphi(x, b)$ bzw. $\varphi(x, a)$ vertauschen

- If q is an heir of $p \in S(M)$ and $\varphi(x, b) \in q$ and $\models \sigma(b)$, then there is some $m \in M$ with $\models \sigma(m)$ and $\varphi(x, m) \in p$.
  Let $q \in S(B)$ be a (co)heir of $p \in S(M)$ and C an extension of B. Then q can be extended to a type $r \in S(C)$ which is again a (co)heir of p.

- A type $p(\overline{x}) \in S_n(B)$ is *definable over C* if for any L-formula $\varphi(\overline{x}, \overline{y})$ there is an $L(C)$-formula $\psi(\overline{y})$ such that for all $\overline{b} \in B : \varphi(\overline{x}, \overline{b}) \in p$ iff $\models \psi(\overline{b})$. p is definable if it is definable over its domain B.
  We write $\psi(\overline{y})$ as $d_p\overline{x}\varphi(\overline{x}, \overline{y})$ to indicate the dependence on p, $\varphi(\overline{x}, \overline{y})$ and the choice of the variable tuple $\overline{x}$.
  In strongly minimal theories all types $p \in S(A)$ are definable.
  A definable type $p \in S(M)$ has a unique extension $q \in S(B)$ definable over M for any set $B \supset M$, namely $\left\{\varphi(x, \overline{b}) \mid \varphi(x, \overline{y}) \in L, b \in B, \mathfrak{C} \models d_p x\varphi(x, \overline{b})\right\}$, and q is the only heir of p.

- A global type which is a coheir of its restriction to a model M is invariant over M
  Let T be strongly minimal, M a model and B an extension of M. Then $\mathbf{tp}(a/B)$ is an heir of $\mathbf{tp}(a/M)$ iff $\mathsf{MR}(a/B) = \mathsf{MR}(a/M)$ (i.e. a and B are geometrically independent).
  ⇒ In strongly minimal theories (actually, in all stable theories!), heirs and coheirs coincide.
  ⇒ In strongly minimal theories types over models have a unique extension of the same Morley rank, i.e., they have Morley degree 1. This is true in all totally transcendental theories.

5.2 Stability

T complete (possibly uncountable). For a formula $\varphi(x, y)$ let $S_\varphi(B)$ denote the set of all φ-types over B; these are maximal consistent sets of formulas of the form $\varphi(x, b)$ or $\neg\varphi(x, b)$ where $b \in B$.

- Let $\varphi(x, y)$ a formula in the language of T.
  φ is *stable* if there is an infinite cardinal λ such that $|S_\varphi(B)| \leq \lambda$ whenever $|B| \leq \lambda$.
  T is stable if all its formulas are stable.
  φ has the *order property* if there are elements $a_0, a_1, ...$ and $b_0, b_1, ...$ such that for all $i, j \in \omega \models \varphi(a_i, b_j)$ iff $i < j$ (the order property is symmetrical)
  $\varphi(x, y)$ has the *binary tree property* if there is a binary tree $(b_s)_{s \in {}^{<\omega}2}$ of parameters such that for all $\sigma \in {}^\omega 2$, the set $\{\varphi^{\sigma(n)}(x, b_{\sigma\restriction n}) \mid n < \omega\}$ is consistent (Where $\varphi^0 = \neg\varphi$ and $\varphi^1 = \varphi$).
  T is stable if and only if it is κ-stable for some κ.

- φ is stable iff $|S_\varphi(B)| \leq |B|$ for any infinite set B iff φ does not have the order property iff φ does not have the binary tree property.

5.3 Definable types

T complete

- A formula φ is stable iff all φ-types are definable
  ⇒ a theory is stable iff all types are definable.
  ⇒ Let T be stable and let $\mathbb{F}$ be a 0-definable class. Then any definable subclass of $\mathbb{F}^n$ is definable using parameters from $\mathbb{F}$
  In stable theories, heirs and coheirs coincide.

- Let p be a global type. If p is definable over A, then p does not divide over A. If T is stable and and p does not divide over a model M, then p is definable over M.
  ⇒ stable theories are simple.
  ⇒ ins stable theories, forking and dividing coincide.

- Let T be stable, p a type over a model M and A an extension of M. Then p has a unique extension $q \in S(A)$ with the following equivalent properties: q does not fork over M, q is definable over M, q is an heir of p and q is a coheir of p.

5.4 Elimination of imaginaries and T^{eq}

- A finite tuple $d \subset \mathfrak{C}$ is called a *canonical parameter* for a definable class $\mathbb{D}$ in $\mathfrak{C}^n$ if d is fixed by the same automorphisms of $\mathfrak{C}$ which leave $\mathbb{D}$ invariant. A *canonical base* for a type $p \in S(\mathfrak{C})$ is a set B which is pointwise fixed by the same automorphisms which leave p invariant.
  ⇒ $\mathbb{D}$ is definable over d, and d is determined by $\mathbb{D}$ up to interdefinability. We write $d = \ulcorner\mathbb{D}\urcorner$, or $d = \ulcorner\varphi(x)\urcorner$ if $\mathbb{D} = \varphi(\mathfrak{C})$. The empty tuple is a canonical parameter for every 0-definable class.

- T *eliminates imaginaries* if any class e/E of a 0-definable equivalence relation E on $\mathfrak{C}^n$ has a canonical parameter $d \subset \mathfrak{C}$
  If T eliminates imaginaries, every definable class $\mathbb{D} \subseteq \mathfrak{C}^n$ has a canonical parameter and every definable type $p \in S(\mathfrak{C})$ has a canonical base.

- Let T eliminate imaginaries. Let A be a set of parameters and $\mathbb{D}$ a definable class. Then: $\mathbb{D}$ is $\mathsf{acl}(A)$-definable iff $\mathbb{D}$ has only finitely many conjugates over A iff $\mathbb{D}$ is the union of equivalence classes of an A-definable equivalence relation with finitely many classes (a finite equivalence relation).

- Let $E_i(\overline{x_1}, \overline{x_2})$, $(i \in I)$ be a list of all 0-definable equivalence relations on n_i-tuples. For any model M of T we consider the many-sorted structure $M^{eq} = (M, M^{n_i}/E_i)_{i \in I}$, which carries the home sort M and for every i the natural projection
  $$\pi_i : M^{n_i} \to M^{n_i}/E_i.$$
  The elements of the sorts $S_i = M^{n_i}/E_i$ are called imaginary elements, the elements of the home sort are real elements.
  The M^{eq} form an elementary class axiomatised by the (complete) theory T^{eq} which, in in the appropriate many-sorted language L^{eq}, is axiomatised by the axioms of T and for each $i \in I$ by
  $$\forall y \exists \overline{x} \pi_i(\overline{x}) \doteq y \qquad (y \text{ a variable of sort } S_i)$$
  and
  $$\forall \overline{x_1}, \overline{x_2} \left(\pi_1(\overline{x_1}) \doteq \pi_2(\overline{x_2}) \leftrightarrow E_i(\overline{x_1}, \overline{x_2})\right)$$

- Elements of $\mathfrak{C}^{eq}$ are definable over $\mathfrak{C}$ in a uniform way. The 0-definable relations on the home sort of $\mathfrak{C}^{eq}$ are exactly the same as those in $\mathfrak{C}$. The theory T^{eq} eliminates imaginaries.
  ⇒ The theory T eliminates imaginaries iff in T^{eq} every imaginary is interdefinable with a real tuple.

- The theory T is $\aleph_1$-categorical(/λ-stable/stable) if and only if T^{eq} is $\aleph_1$-categorical(/λ-stable/stable)
- T *eliminates finite imaginaries* if every finite set of n-tuples has a canonical parameter. T has *weak elimination of imaginaries* if for every imaginary e there is a real tuple c such that $e \in \mathsf{dcl}^{eq}(c)$ and $c \in \mathsf{acl}(e)$
  T eliminates imaginaries iff it has weak elimination of imaginaries and eliminates finite imaginaries
  Let T be strongly minimal and $\mathsf{acl}(\varnothing)$ infinite. Then T has weak elimination of imaginaries.
  A totally transcendental theory in which every global type has a canonical base in $\mathfrak{C}$ has weak elimination of imaginaries.

6 Results

- DLO is $\aleph_0$-categorical (and thus complete) and has QE.
- ACF has QE.
  $\mathsf{ACF}_{p/0}$ are κ-categorical for $\kappa > \aleph_0$ (and thus complete), κ-stable for all κ, strongly minimal and eliminate imaginaries.
- RCF and DCF_0 have QE and are complete.
- DCF_0 has QE, is complete and eliminates imaginaries.
- The theory of K-vector spaces $\mathsf{Mod}(K)$ is κ-categorical for $\kappa > |K|$.
- the theory of all infinite K-vector spaces has QE and is complete and strongly minimal.
  The theory of infinite $\mathbb{F}_q$-vector spaces is $\aleph_0$-caterogical.
- Infset has QE, is $\aleph_0$-caterogical, strongly minimal and is complete.
- RG is $\aleph_0$-caterogical, simple (but not stable) and the model companion of Graph.
- The theory of any R-module is stable.